A Treatise on the Preponderance of Designs Over Historic and Measured Snowfalls, or No Two Snowflakes Are Alike: Considerations About the Formation of Snowflakes and the Possible Numbers and Shapes of Snowflakes

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"The reason for this macroscopic six-fold symmetry, which so exercised Kepler, is now readily explained in terms of the atomic architecture of ice, but the variety and change of the crystal shape has always been a puzzle and still awaits a complete explanation."

— B.J.Mason in A Commentary on Kepler's essay ON THE SIX-CORNERED SNOWFLAKE

ABSTRACT

We use a generalized geometric model for the shape of snowflakes to approximate the finite number of possible unique snowflake configurations. Using historic geologic and meteorological data we approximate the number of snowflakes created throughout history. Assuming the development of snowflakes to be subject to guided random molecular movements, we calculate the probability of duplicate configurations. The factorial estimate of possible snowflake configurations overwhelms the countable linear quantity of snowflakes. We show, even with very conservative estimates, that except for extremely small snowflake fragments, no two snowflakes are identical. Thus, at the macro-optical scale, no two snowflakes are alike.

PREMISE

Simply stated, we determine from a general model for snowflakes how many possible snowflake configurations may exist. Using geologic and meteorological data we approximate the number of snowflakes created throughout history. If the total number of actual snowflakes exceeds the possible number of snowflake configurations, we deduce that at least two snowflakes throughout history have been identical. If this is so, we will be able to determine how often identical snowflakes fall. If the total number of possible snowflake configurations greatly exceeds the total number of snowflakes that have fallen throughout history, we must concede that it is improbable that two identical snowflakes have ever fallen.

To evaluate this premise, we sought to answer these two questions:

1. How many possible snowflake configurations are there?
2. How many snowflakes have fallen?

The answer to the first question will be the result of our modeling. We develop a model that approximates the shape of a snowflake. We limit our efforts to one form of snowflake; the dendritic plate.

The answer to the second question requires approximations. We chose such to maximize our result. Thus, for the total number of snowflakes that have fallen, our approximation will be higher than the historical value.

To determine how many snowflakes have fallen we need to know, how much snow has fallen and the average size of a snowflake. Measurements of snowfall are made locally not globally. Any amount of snow that we project to have fallen on the earth will be, at best, a guess.

SNOWFLAKE MODEL

Combinations and configurations

This is a model for a flat hexagonal snowflake and incorporates dendritic plates. Several simplifications make this model workable and yet remain consistent with what is known about real snowflakes. The model begins, as any good model, with actual snowflakes. (3)

Our model applies to dendritic plates:

We assume snowflakes have:

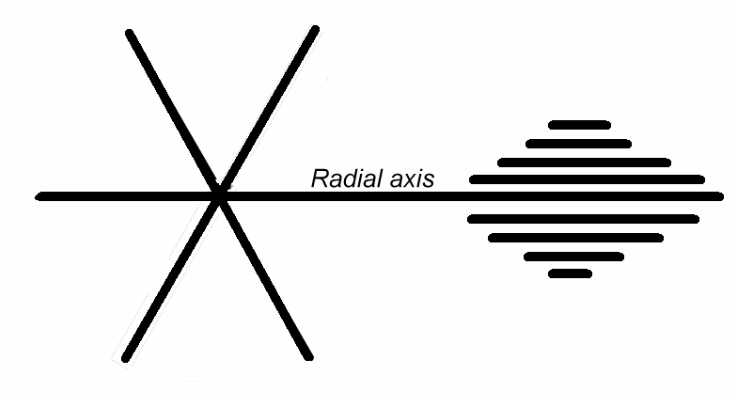
- Hexagonal radial symmetry (six-sided snowflakes),
- Two-dimensional configuration (flat snowflakes), and
- The simplest unit of an ice crystal is a hexagonal ring composed of six water molecules.

We know that real snowflakes have several documented forms and, although, these may be hexagonal in nature, they are not strictly two-dimensional. They have thickness and the basic building blocks of any ice crystal may be far more complex than a ring of only six water molecules. Thus, any model we develop will underestimate the number of possible snowflake configurations.

Each of the above assumptions carries several corollaries that shape our model. Hexagonal radial symmetry implies that:

a. Each of the six arms are identical to each other.
   i. One sixth of the total number of units is used to compose each of the six arms.

b. Each arm is bilaterally symmetric.
   (It is the same on each side of its radial axis.)
   i. There is a centerline of units. Assume the centerline (radial axis) to be the longest line of each arm.
   ii. The sidelines (parallel to the centerline) are paired, one on either side of the centerline.
   iii. The sidelines are made up of many segments of lines each shorter than the centerline.
   iv. The length of each sideline must be at least three units shorter than the next sideline closer to the centerline and at least three units longer than the next sideline farther from the centerline.

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                              -----       -------------
                         -------   ------------      ----------    sideline (3 segments)
                      -------   -----  -      -----------    ---------    sideline
                     ----------------------------------------------   sideline
Radial axis    -----------------------------------------------------------  one of six arms
                     ----------------------------------------------
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We know that the most common form of ice is a tetrahedral structure. Idealized hexagonal rings are found within the tetrahedral form. Close examination of the basal plane of an ice crystal reveals it is composed of a tiling of hexagonal plates each with a diameter of 5.5 angstroms, 4.5 angstroms across the tiling diameter or minor chord and 2.75 angstroms between adjacent oxygen atoms. (1)(5)

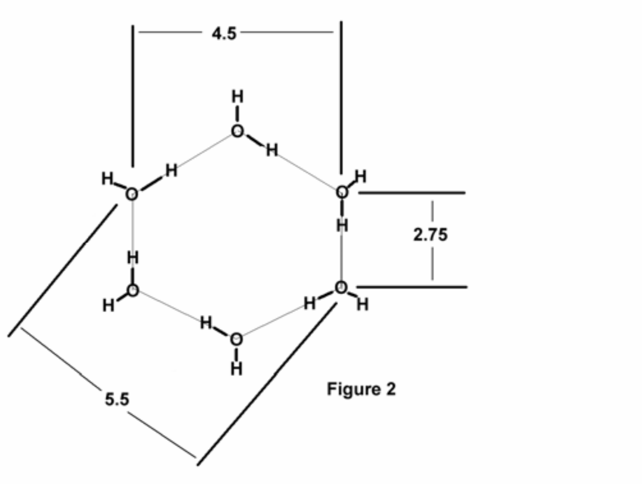

Figure 2

Using a simple idealized hexagonal ice crystal as a building block, we can approximate how many such blocks or hexagonal tiles must accumulate to form a single axis arm of a snowflake.

Ordinary snowflakes vary in size from 0.05 cm. to 1.8 cm. in diameter. A reasonable midrange value for the average radius of a snowflake is 0.66 cm. Thus, a radius axis of an average snowflake is

$$\frac{6.6 \times 10^{-1} \text{ cm. (radius axis length)}}{4.5 \times 10^{-8} \text{ cm. (hexagonal unit length)}} = 1.5 \times 10^{7} \text{ hex units}$$

1.5×10^{7} units is the number of hexagonal units along any one axis.

The model calls for the axis and any of the sidearms to be at most filled with hexagonal units and in most cases present with gaps in the linear tiling. We may think of any axis or sidearm of actual hex units to be represented by a binary number (hex unit present = 1, hex unit absent = 0.)

A sidearm three units in length would have

$2^3 - 1 = 7$ possible configurations of ice hex units

111, 110, 101, 100, 011, 010, 001

The null configuration of no hex units (000) is not possible because the resulting snowflake would lack any cohesion along this null line.

A radius axis or sidearm with 1.5×10^{7} units would have

$(2^{1.5 \times 10^{7}} - 1)$ possible permutations.

For a configuration consistent with our model we have, where $N = 2^{1.5 \times 10^{7}}$

$2^{(N-3)}-1$ is the number of possible permutations for the next smaller sidearm(s). Then it follows

$$2^{(N)}-1 \cdot 2^{(N-3)}-1 \cdot 2^{(N-6)}-1 \cdot 2^{(N-9)}-1 \ldots = \prod_{X=0}^{N/3} 2^{(N-3X)}-1$$

Equals the number of possible permutations and combinations of all sidearms, i.e. the number of possible unique snowflakes. The multiplication of exponential quantities may be expressed as the sum of these exponents. We may simplify this series as

$$2^{\sum (N-3X)} - 1 = 2^{\sum_{0}^{N/3} (3X+1)} - 1 = 2^{3\sum_{0}^{N/3} (X) + \sum_{0}^{N/3} (1)} - 1 =$$

$$\prod_{X=0}^{N/3} 2^{(N-3X)}-1 \quad \text{or} \approx 2^{3.8 \times 10^{13}} \quad \text{or} \approx 10^{10^{13}}$$

That's a one followed by ten trillion zeros, as the possible number of unique snowflake configurations.

No one to date has shown why snowflakes (here we are discussing only the formation of plates, specifically dendritic plates) demonstrate a remarkably precise hexagonal radial symmetry. The currently accepted explanation for this is that as snowflakes form they travel along unique random paths that expose the developing snowflake to micro environmental fluctuations. Crystalization occurs in response to these local conditions. The nature of water molecules leads to the familiar hexagonal crystals. Thus, the unique path taken by a developing snowflake determines its final configuration. (4)(9)

Another possible contributing factor for the randomness we see in snowflake formation is the presence of a quasi-liquid phase as part of a developing snowflake. (7)

We postulate another possible explanation for these guided random configurations of snowflakes; the development of an emerging pi-obital like electron cloud. This elaborate pi-orbital develops as the crystal grows creating sites favored for the attachment of the next water molecules. Such pi-orbitals are found in large organic molecules (i.e. planar benzene constructs) as well as several planar inorganic crystalline forms. We consider the formation of ice pi-orbitals, much like those of benzene molecules. The individual water (ice) hex plates may each have such a pi-orbital and as they are joined to form a larger plate this orbital expands to 'guide' the future formation of the snowflake.

Regardless of the exact cause for the observed symmetry, we find it to be guided (following a symmetric pattern) and random (each crystalline growth point may or may not affix a water molecule or molecular complex.

The physical implications of this model lead us to claim that there are

1.5×10^7 hex units in a single radial axis; 3.8×10^{13} hex units in a complete radial arm (a radial axis and all sidearms.) There are six radial arms per each snowflake and six water molecules for each ice hex unit. These figures give us

(3.8×10^{13} hex units per arm) × (6 arms per snowflake) × (6 H_2O per hex unit)

= 1.4×10^{15} H_2O molecules in a model snowflake.

This is based upon a two-dimensional model. Actual snowflakes, as far as they might inform this model, have thickness. By adding several layers of parallel planes of ice above and below the central plane of this model, we see that we approach the measured mass of observed snowflakes. We also appreciate that each added plane of ice multiples our estimate of possible snowflake configurations to beyond the already predicted astronomical numbers.

Throughout the development of this model we've assumed that there is an equivalence of the probability of any possible snowflake configuration. In other words, the probability of any one snowflake configuration is the same as any other snowflake configuration. This may not be a valid assumption. Indeed, there may be favored snowflake configurations. Thus, the final form of any actual snowflake may not be the product of strictly random development. If this is true, our estimate of all possible snowflake configurations is an over-estimate of any real quantity

Photographs show that some real snowflakes are asymmetric. Whether this asymmetry is evident while a snowflake develops or induced owing the capture processes, symmetry prevails. We can argue *ad nauseum* about the definition of 'alike' or 'identical.' We may also accept the concept of 'for all practical purposes.'

SNOWFALL

There are 1.4×10^{15} H_2O molecules or 4.2×10^{-8} gms of H_2O in a model snowflake. If we approximate the average water density of fresh snow to be 12%, this is equivalent to saying that 1.0 cm^3 of snow is equal to 0.12 gms of H_2O.

Using an estimate of total snowfall over the earth's surface per year of 3.3×10^{21} cm^3 (see Appendix A) and our model, we obtain 9.6×10^{31} model snowflakes produced worldwide each year.(8) If we consider data for actual snowflakes we obtain 6.6×10^{28} actual snowflakes produced worldwide each year. (2)(6)

The earth is by currently available measures about 4.5×10^{9} years old. Thus, for our model we obtain 4.3×10^{39} snowflakes produced throughout history and for actual snowflakes we obtain 3×10^{38} as the number of historic snowflakes. Both numbers are close to each other and we use this closeness as a validation of the accuracy of the model.

From our model we obtain a value of 10^{13} for the number of possible snowflake configurations. (We have no evidence that there are preferred configurations.) We also estimate that only 10^{40} actual snowflakes have been produced throughout the history of the earth. (Of course there may have been great variations in seasonal snowfall throughout history. By maintaining a consistent yearly snowfall, we may over-estimate or under-estimate this value by a few orders of magnitude. With the current rate of snow production on earth it will be many trillions upon trillions of years before the production of two identical snowflakes may be considered a possibility. For now we may assume that there simply has not been enough snow to produce two identical snowflake configurations.

APPENDIX A

Northern Hemisphere Yearly Snowfall Totals (8)

Band	Area (cm^2)($x10^{16}$)	Avg Snowfall (cm)	Snowfall (cm^3) ($x 10^{19}$)
$90^o – 86^o$	0.627	2888	1.81
$86^o – 82^o$	1.866	3142	5.86
$82^o – 78^o$	3.068	3584	11.0
$78^o – 74^o$	4.212	3975	16.7
$74^o – 70^o$	5.277	3800	20.1
$70^o – 66^o$	6.233	3546	22.1
$66^o – 62^o$	7.081	3618	25.6
$62^o – 58^o$	7.766	5207	40.4
$58^o – 54^o$	8.328	3698	30.8
$54^o – 50^o$	8.708	2029	17.7
$50^o – 46^o$	8.932	589	5.26
$46^o – 42^o$	8.959	490	4.39
$42^o – 38^o$	8.740	140	1.22
$38^o – 34^o$	8.640	28	0.24
$34^o – 30^o$	8.011	30	0.24
$30^o – 26^o$	7.194	56	0.40

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