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A Technique to Estimate Snow Depletion Curves From Time-Series Data Using the Beta Distribution

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EXTENDED ABSTRACT

Snow depletion curves, required as input to many snowmelt models, relate the time-varying snow-covered fraction, *S(t)*, to the cumulative melt energy received, *M(t).* In hydrologic models, snow water equivalent (SWE, particularly the mean value) is generally the most important variable in mass accounting; however, *S(t)* is more easily observed (by remote sensing, for example). In atmospheric models, *S(t)* exerts a strong control on land–atmosphere coupling. Generally, the coarse resolution of model grids or response units requires a sub model relating *S(t)* to areal-average SWE.

Depletion curves often express snow-covered fraction directly as a function of cumulative melt, in the form *S(M).* Under simplifying assumptions, *S(M)* can be derived from the end-of-winter frequency distribution of snow water equivalent (SWE) (Martinec 1972, Liston 1999). Usually, however, this distribution is unknown, and *S(M)* must be inferred from sparse observational data of *S(t)* and estimated *M(t).* We make the simplifying assumption that all disappearance of snow is due to melt, and define the following variables: $Z(\mathbf{x})$ = SWE [cm] at end of winter (start of melt), a random variable in space, described by a probability density function *f*(*Z*); *M(t)* = Cumulative potential melt energy, expressed as melt depth [cm], monotonically increasing in time; and *S(t)* = areal fraction snow covered [dimensionless], monotonically decreasing in time. Assuming that *M(t)* is uniform in space, snow cover depletion *S(t)* as a function of *M(t)* is equal to the complement of cumulative distribution function (cdf) for end-of-winter $Z(\mathbf{x})$:

$$\begin{aligned} S(t) &= P[Z > M(t)] = 1 - P[Z \le M(t)] \\ &= 1 - F[M(t)] = 1 - \int_{-\infty}^{M(t)} f(Z)\,dZ \end{aligned} \qquad (1)$$

Strictly speaking, *f(Z*) is a mixed distribution, defined by a discrete probability of P(*Z*=0), and a conditional density function, given that *Z* is nonzero.

Given an incomplete time series of $S(t_i)$, i = 1 to *n*, from remote sensing, the goal is to infer or back-calculate end-of-winter P(*Z*=0) and *f*(*Z*). Our approach is to use a model to build a time series of *M(t);* match $M(t_i)$ to the days of $S(t_i)$ observations; then select an appropriate density function and fit its cdf to the *n (M,S)* observations using nonlinear regression.

As a parametric model for the spatial distribution of point SWE, the normal probability density function (pdf) is well studied, but problematic: it is not bounded at zero, whereas *SWE* cannot take negative values, and its symmetry is not always appropriate. The lognormal pdf is popular for SWE (*e.g.* Shook & Gray 1993, Faria et al. 2000); it is bounded at zero and has an asymmetric

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mound shape that has been found to match empirical SWE histograms well in some cases. The lognormal pdf is defined to positive infinity; like the normal pdf, it is concisely defined by two parameters; and it is always positively skewed.

As a model for end-of-winter SWE distribution, the Beta distribution has the advantages of upper and lower limits and a flexible shape. The classic Beta pdf is defined on 0 to 1, and has two shape parameters, α and β. Adjusting these two parameters, one can obtain a family of pdfs with flexible skewness, and flexible shape at the origin and upper limit. The independent variable can be transformed to allow an arbitrary minimum (*A*) and maximum (*B*).

As an example, we have used the Beta pdf to develop depletion curves for one elevation zone of the Rio Grande basin above Del Norte, using National Operational Hydrologic Remote Sensing Center operational products for *S(t)* at 1100 m resolution. For *M(t)*, we apply a degree-day method (Martinec 1994); this method is known to be a simplification of melt processes, but allows us to illustrate the method. Better physically based estimates of *M(t)* will improve the results. Temperature and snow density measurements used in the degree-day estimate of *M(t)* were obtained from Natural Resources Conservation Service SNOTEL stations in the watershed. The lower bound of $Z(\mathbf{x})$ was fixed at A=0, leaving four parameters to be estimated: α, β, B, and P(Z=0). Automatic nonlinear regression ("Solver" in Microsoft Excel) was used to do the curve fitting. The objective function to be minimized consisted of a weighted sum of squared errors (predicted – observed), where the weighting function for each date was determined by the cloud-free fraction of the satellite image.

Both the application of the Beta distribution and the use of automatic nonlinear regression to infer a depletion curve from sparse data are novel. Therefore, we were interested in exploring the uniqueness of the solutions and sensitivity to constraints imposed in "Solver".

In a first attempt, the Beta pdf parameter α was fixed at 3 to enforce a zero slope at the origin. The automatically curve-fit *S(M)* and the parameters of the Beta pdf are given in Figure 1a; the corresponding *S*(t) appears below it (Fig. 1b); both of these curves appear reasonable. However, the parameters obtained by the automatic curve fit define a pdf that lies far outside the range expected for SWE; the "maximum SWE," B, determined by the optimization is unrealistically high (over 400,000 cm) and no probability density is visible in Figure 1(c). Clearly, the constraint of α = 3 forced unrealistic values of β and *B* in this optimization. Analysis of the solution space (not shown) reveals a very slow descent of the objective function along a line of increasing β and *B* when α = 3, as well as several local, physically irrational minima.

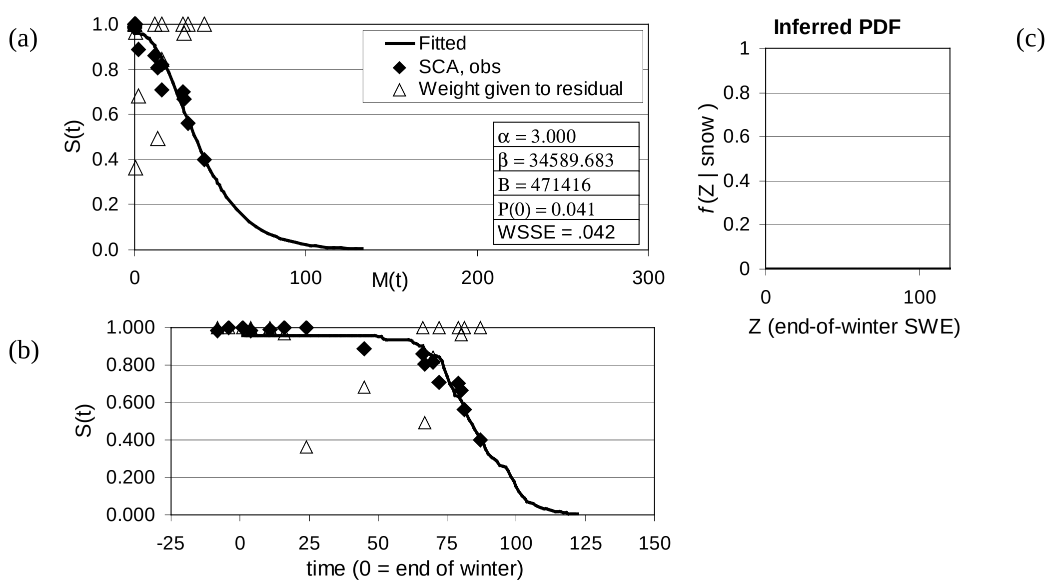

Figure 1. Results of automatic curve-fitting for example problem, with constraint α = 3: (a) snow-covered area vs. cumulative melt; (b) snow-covered area vs. time; (c) inferred conditional probability density function for end-of-winter SWE.

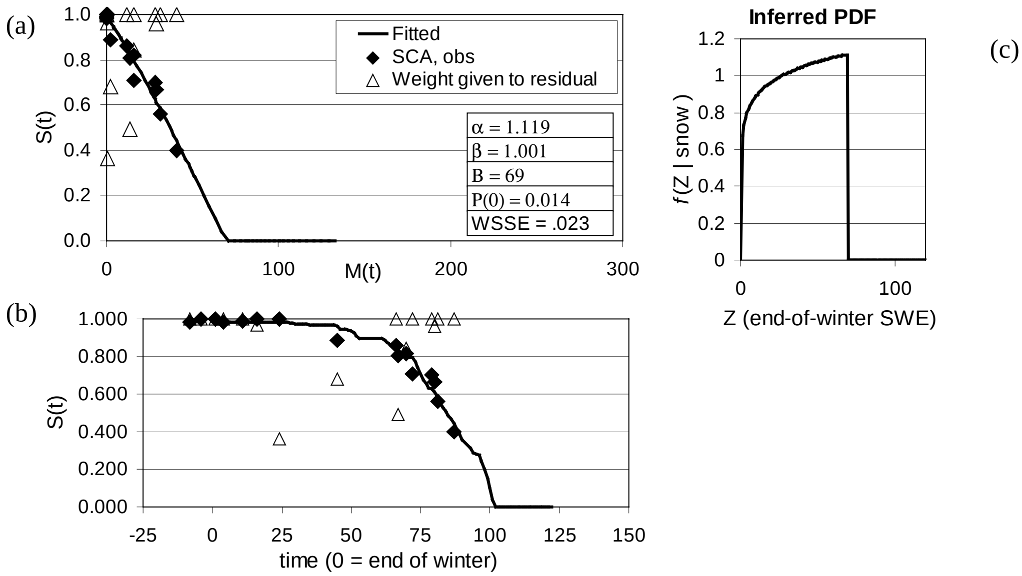

Figure 2. As in Fig. 1, but with constraints $\alpha > 1$ and $\beta > 1$.

In a second attempt, α and β were simply constrained to be greater than 1. The results of this optimization are shown in Figure 2. The range of the pdf, with B = 69, is much more realistic than in Figure 1, and the value of the objective function somewhat lower. However, the shape of the inferred pdf of $Z(\boldsymbol{x})$ is not rational; physically, the pdf should be continuous at $Z = B$; the graph shows an unrealistic difference between $P(Z<69-\delta)$ and $P(Z>69+\delta)$, where δ is small.

One way to force this continuity at the upper bound is to constrain β to be greater than or equal to 2 in the Beta pdf. The third curve-fitting attempt (Figure 3) shows the effect of this set of constraints. The objective function is small, compared to the first two attempts. Both the depletion curves and the inferred pdf appear rational. Analysis of the solution space (not shown) reveals a unique minimum at the $[\alpha,\beta,B,P(Z=0)]$ point located by the optimization.

Conclusions and Recommendations. The spatial pdf of end-of-winter SWE is actually a mixed distribution with a discrete probability of no snow, and a conditional density function *f*(SWE| snow is present). The Beta pdf provides a variety of shapes and scales for the conditional distribution of SWE. The free parameter space can be reduced to four parameters: the shape parameters α, β; the maximum SWE, B, and the snow-free area at end of winter, $P(Z=0)$.

Some subjectivity is involved in using an automatic optimization routine to curve-fit the Beta cdf due to: the potential non-uniqueness of solution and corresponding sensitivity to the initial guess in optimization routine, and the sensitivity to constraints. Some minima of the objective function may represent irrational solutions. A user should examine *M*–*S*, *t*–*S* and the inferred pdf *f*(SWE) graphically to assure a rational solution consistent with hydrologic judgment. With attention to appropriate constraints and rationality of the solution, local optima of the objective function correspond to physically believable density functions. As a topic for future study, the pdfs inferred from this approach need to be checked against observed distributions of SWE over large areas, where such data are available.

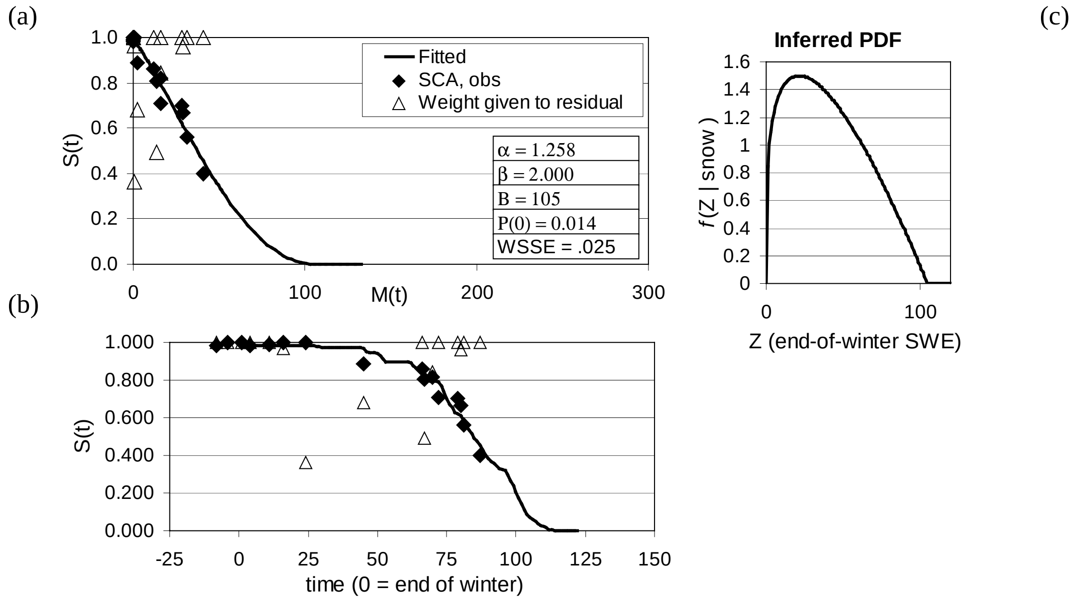

Figure 3. As in Figs. 1 and 2, but with constraints $\alpha > 1$, $\beta \geq 2$.

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